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Going-Down Pairs of Commutative Rings

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Background

GD : going-down property (of ring extensions)

INC : incomparability property

LO : lying-over property

GU : going-up property

$Spec(R)$: prime ideals of R with the Zariski topology

$Min(R)$: minimal prime ideals of R

$\dim(R)$: Krull dimension of R

$t.d._R(T)$: transcendence degree of the quotient field of T over the quotient field of R

Def: A ring extension $R \subseteq T$ is **open** if the contraction map $\text{Spec}(T) \rightarrow \text{Spec}(R)$ given by $P \mapsto P \cap R$ is an open topological map.

Def: Let $\mathcal{P}$ be a property of ring extensions and let $R \subseteq T$ be a ring extension. Then (R, T) is a **$\mathcal{P}$ -pair** if $A \subseteq B$ satisfies $\mathcal{P}$ for all ring extensions $A \subseteq B$ such that $R \subseteq A \subseteq B \subseteq T$.

LO-pairs, *GU*-pairs, and *INC*-pairs have been studied previously in the literature.

Def: Let $\mathcal{P}$ be a property of ring extensions. A domain R is a $\mathcal{P}$ -**domain** if for all overrings S of R , $R \subseteq S$ satisfies $\mathcal{P}$.

Ex: If a domain R is Prüfer or has $\dim(R) \leq 1$, then R is a GD -domain.

Ex: Let R be a Noetherian domain. Then R and all its overrings are open-domains $\iff R$ is semilocal and $\dim(R) \leq 1$ (Papick, 1976).

Def: We say (R, T) is a $\mathcal{P}$ -**domain pair** if S is a $\mathcal{P}$ -domain, for all S such that $R \subseteq S \subseteq T$.

Ex: If R is a Prüfer domain and K is its quotient field, then (R, K) is a GD -domain pair.

Ex: If $k \subseteq K$ is an algebraic field extension, then (k, K) is an open-domain pair.

Ex: Let R be Noetherian and k the quotient field of R . Then (R, k) is an open domain pair $\iff R$ is semilocal and $\dim(R) \leq 1$. (Restatement of above example of Papick)

Def: A ring R is a $\mathcal{P}$ -**ring** if $\forall P \in \text{Spec}(R)$, R/P is a $\mathcal{P}$ -domain. (For $\mathcal{P} = GD$, Dobbs 1996)

Ex: $\dim(R) \leq 1 \Rightarrow R$ is a GD -ring. Other examples of GD -rings include pseudo-valuation rings, chained rings, and finite products of GD -rings.

If R is a domain, then R is a GD -ring (resp., open-ring) $\iff R$ is a GD -domain (open domain).

Def: We say (R, T) is a **$\mathcal{P}$ -ring pair** if S is a $\mathcal{P}$ -ring, $\forall S \ni R \subseteq S \subseteq T$.

Proposition 1 *The following are equivalent:*

1. (R, T) is a GD-ring pair (resp., open-ring pair);
2. For each $Q \in \text{Spec}(T)$,
 $(R/(Q \cap R), T/Q)$ is a GD-domain pair (open domain pair);
3. For each $Q \in \text{Min}(T)$,
 $(R/(Q \cap R), T/Q)$ is a GD-domain pair (open domain pair).

Transcendence degree results

Lemma 2 *If k is a field contained in a domain T , then (k, T) is a GD-domain pair $\iff t.d._k(T) \leq 1$.*

Theorem 3 *If $\dim(R) = 0$, then the following are equivalent:*

1. (R, T) is a GD-ring pair;
2. $t.d._{R/P \cap R}(T/P) \leq 1, \forall P \in \text{Spec}(T)$;
3. $t.d._{R/P \cap R}(T/P) \leq 1, \forall P \in \text{Min}(T)$.

Lemma 4 *If k is a field contained in a domain T , then (k, T) is an open-ring pair $\iff k \subseteq T$ is an algebraic field extension.*

Theorem 5 *If $\dim(R) = 0$, then the following are equivalent:*

1. (R, T) is a open-ring pair;
2. $t.d._{R/P \cap R}(T/P) = 0, \forall P \in \text{Spec}(T)$;
3. $t.d._{R/P \cap R}(T/P) = 0, \forall P \in \text{Min}(T)$;
4. (R, T) is an INC-pair;
5. T is an integral extension of R .

What if $\dim(R) \geq 1$?

Proposition 6 $R[X]$ is a *GD-ring* $\iff \dim(R) = 0$.

Corollary 7 If (R, T) is a *GD-ring pair* and $\dim(R) > 0$, then $R \subseteq T$ is an algebraic ring extension.

The converse of this corollary fails: consider $(\mathbb{Z}[X], \mathbb{Z}[X][\sqrt{2}])$.

GD-ring pairs need not be *INC*-pairs, even when $\dim(R) > 0$.

Proposition 8 If $\dim(R) \leq 1$ and (R, T) is an *INC-pair*, then (R, T) is a *GD-ring pair*.

For $\dim(R) \geq 2$, *INC*-pairs need not be *GD-ring* pairs.

Proposition 9 *If (R, T) is an open-ring pair, then (R, T) is an INC-pair.*

Proposition 10 *Let (R, T) be a normal pair of domains (i.e. A is integrally closed in $B, \forall R \subseteq A \subseteq B \subseteq T$). Then (R, T) is a GD-domain pair (resp., open domain pair) $\iff R$ is a GD-domain (open domain).*

Relation between $\mathcal{P}$ -ring pairs and $\mathcal{P}$ -pairs

If R is a GD -domain, $R \subseteq T$ satisfies GD for all domains T containing R . (Dobbs-Papick, 1976)

So (R, T) GD -domain pair $\Rightarrow (R, T)$ GD -pair.

If R is a open domain, $R \subseteq T$ is open for all domains T containing R . (Papick, 1976)

So (R, T) open domain pair $\Rightarrow (R, T)$ open pair.

Proposition 11 $(\mathbb{F}_p, \mathbb{F}_p(X))$ is an *INC-domain pair* that is not an *INC-pair*.

Also, R a *GD-ring* does **not** imply $R \subseteq T$ satisfies *GD* for all rings T containing R . (Dobbs, 1996)

Theorem 12 Let $\mathcal{P} \in \{GD, open\}$. Then R a $\mathcal{P}$ -ring $\Rightarrow R \subseteq T$ satisfies $\mathcal{P}$ for all T with a unique minimal prime.

Corollary 13 Let $\mathcal{P} \in \{GD, open\}$. If T is a ring with a unique minimal prime and (R, T) is a $\mathcal{P}$ -ring pair, then (R, T) is a $\mathcal{P}$ -pair.

Are $\mathcal{P}$ -pairs always $\mathcal{P}$ -ring pairs?

Ex: Let $k \subseteq K$ be an algebraic field extension and T a ring that is *not* a GD -ring. Then $(k \times T, K \times T)$ is a GD -pair and an open-pair that is not a GD -ring pair (or an open-ring pair).

Ex: Let $L \subset K$ be an algebraic extension of distinct fields. Let $T = K[[X, Y]]$ and $R = L + XT + YT$. Then (R, T) is a GD -pair of domains that is not a GD -domain pair.

Flat pairs and flat-domain pairs

Def: A ring extension $R \subseteq T$ is **flat** if T is a flat R -module.

A domain R is a flat-domain $\iff R$ is a Prüfer domain (Richman, 1965)

Therefore, we will refer to a flat-domain pair as a “Prüfer domain pair.”

Proposition 14 *Let R be a domain contained in a field K . Then the following are equivalent:*

1. *(R, K) is a flat pair;*

2. *(R, K) is a Prüfer domain pair;*

3. **either**

(a) $R \subseteq K$ is an algebraic field extension;

or

(b) R is a Prüfer domain and K is the quotient field of R .

Proposition 15 *Let $R \subseteq T$ be domains with R Prüfer. Then the following are equivalent:*

1. (R, T) is a flat pair;
2. (R, T) is a Prüfer domain pair;
3. **either**
 - (a) $R \subseteq T$ is an algebraic field extension;**or**
 - (b) T is an overring of R .