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Internal Design of Uniform Shear Rate Dies

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Abstract

Design equations of uniform shear rate dies for power law non-Newtonian fluids are presented together with their derivations, and flow properties of such dies are examined. In an attempt to decrease the limitations of such dies, modifications of design equations are given and results are discussed.

Introduction

A uniform shear rate die is one for which the shear rate is constant at any given point along the wall. This requirement makes the die insensitive to a fluid's power law exponent. Thus, once such a die is constructed for a certain fluid, all other fluids, which follow the power law can be extruded uniformly through the same die. This represents one of the main advantages of a uniform shear rate die. Another unique advantage of these dies is that their flow residence times are constant; dependent only upon the characteristics of the extruded material. For an internally consistent extrudate, every particle will take the same amount of time to flow through the die, no matter the course. The combination of these two properties makes such a die suitable for extrusion of shear rate sensitive materials, and also capable of extruding several power-law materials effectively.

This paper is organized into three sections. In section one we present the mathematical derivations that lead to the design equations of a uniform shear rate die known as the Winter Die, examine the flow properties of this die, and describe this die's limitations. It is shown that the uniform shear rate requirement puts certain restrictions on the dimensions of the Winter Die. These restrictions make the die unsuitable for extrusion of degradable material or for operations that involve high pressure-drops. In section two we examine different models, which are used to alter the design of the Winter Die, and effectively lessen its limitations. Finally, in section three we give several concluding remarks.

1 Design Equations and Flow Properties

Throughout this paper, it is assumed that the material viscosity-shear rate relationship can be described by the power law,

$$\eta = k\gamma^{(n-1)} \tag{1.1}$$

which may be written as

$$\eta = \eta_0 \left(\frac{\gamma}{\gamma_0}\right)^{n-1} \tag{1.2}$$

where $\gamma \geq \gamma_0 \geq 0$. In (1.2) η, η_0, n, and γ represent viscosity, zero viscosity, the power law material index, and shear rate, respectively.

An example of a uniform shear rate die is the Winter Die. We will confine ourselves to the derivation of design equations for the Winter die with a rectangular cross section manifold of uniform width (see figure 1.1).

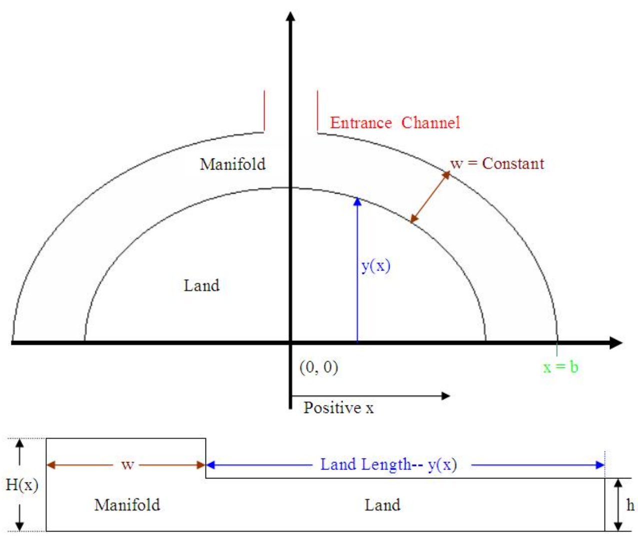


Figure 1.1: Top View (top); Cross-Section (bottom)

Let v_m be the average velocity through the manifold, v_s the average velocity through the land or slit, $Q(x)$ the flow rate through the die, a the arc length of the manifold, and P the pressure.

The condition of uniform shear rate gives

$$Q(x) = (b-x)hv_s = wH(x)v_m \tag{1.3}$$

which states that the rate of material passing through the manifold at the position x is equal to the rate of material exiting the system between x and b.

The pressure gradients in the manifold, dP/da, and in the land, dP/dy, are related by

$$\frac{dy}{dx} = -\left[\left(\frac{dP/dy}{dP/da}\right)^2 - 1\right]^{-1/2} \tag{1.4}$$

By examining the power law, we find that the shear stress, τ, is proportional to the shear rate by the proportionality constant known as viscosity. That is,

$$\tau = \eta_0 \gamma_0 \left(\frac{\gamma}{\gamma_0}\right)^n \tag{1.5}$$

The relationship between the shear stress and the pressure gradient along the y-axis can be understood by recognizing that the shear rate is a function along the coordinate axis orthogonal to the die wall. The die is shown in Cartesian three-space below

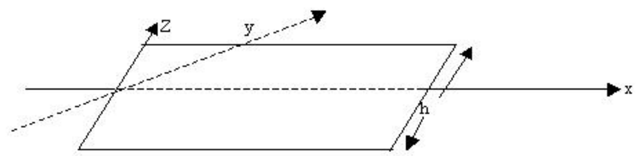


Figure 1.2:

The shear rate is related to the pressure gradient by

$$\frac{\partial \tau}{\partial z} = \frac{dP}{dy} \tag{1.6}$$

which, upon integration and using the boundary condition $\tau(0) = 0$ (wall adhesion) gives

$$\tau(z) = \frac{dP}{dy} z \tag{1.7}$$

Using equation (1.7) together with

$$\gamma \equiv -\frac{dv_y}{dz} \tag{1.8}$$

implies

$$\tau_w = -\eta_0 \gamma_0^{1-n} \left(\frac{dv_y}{dz}\right)^n \tag{1.9}$$

Using $\gamma = \gamma_w$ when $z = h/2$, and from Michaeli [64], we have

$$\gamma_w = \frac{2(m+2)Q(x)}{bh^2} \tag{1.10}$$

from which $m = 1/n$. From equation (1.3) it follows that

$$\gamma_w = \frac{(2 + 1/n)v_s}{h/2} \tag{1.11}$$

Similarly, the shear rate at the wall of the manifold can be shown to be

$$\gamma_w = \frac{(2+1/n)v_m}{H(x)/2} \tag{1.12}$$

Now, since we have stated that the shear rate remains constant throughout the die, equations (1.11) and (1.12) imply

$$\frac{v_m}{v_s} = \frac{H(x)}{h} \tag{1.13}$$

which upon using (1.3) gives

$$\frac{v_m}{v_s} = \frac{b-x}{w}\frac{h}{H(x)} \tag{1.14}$$

From (1.13) we acquire

$$\frac{H(x)}{h} = \frac{b-x}{w}\frac{h}{H(x)} \tag{1.15}$$

which in turn leads to,

$$H(x) = \left(\frac{b-x}{w}\right)^{1/2} h \tag{1.16}$$

Equation (1.16) describes the manifold height as a function of x.

Now, we focus our attention on the pressure in the die, particularly on the pressure gradients in the manifold and land. By rewriting equation (1.7) as

$$\frac{\tau(z)}{z} = \frac{dP}{dy} \tag{1.17}$$

and using equation (1.5), it follows that

$$\frac{dP}{dy} = \frac{\eta_0\gamma_0\left(\frac{\gamma}{\gamma_0}\right)^n}{z} \tag{1.18}$$

Since $\gamma = \gamma_w$ at $z = h/2$ we have

$$\frac{dP}{dy} = \frac{2\eta_0\gamma_0\left(\frac{\gamma_w}{\gamma_0}\right)^n}{h} \tag{1.19}$$

Combining Equations (1.18) and (1.19) with (1.16) and the given equation

$$\frac{dP}{da} = \frac{2\eta_0\gamma_0\left(\frac{\gamma_w}{\gamma_0}\right)^n}{H(x)}$$

it follows that

$$\left(\frac{dP/dy}{dP/da}\right)^2 = \frac{b-x}{w} \tag{1.20}$$

Substituting the right side of equation (1.20) in equation (1.4) gives

$$\frac{dy}{dx} = -\left[\left(\frac{b-x}{w}\right) - 1\right]^{-1/2} \tag{1.21}$$

which upon integration gives

$$y(x) = 2w\left[\left(\frac{b-x}{w}\right) - 1\right]^{1/2} \tag{1.22}$$

Equations (1.16) and (1.22) are called the design equations for the winter die.
Remark: It follows from (1.22) that the maximum length of the land is given by

$$y_{\max} = 2w\left[\left(\frac{b}{w}\right) - 1\right]^{1/2} \tag{1.23}$$

Also, from (1.22), it follows that as b grows, the length of the land becomes large. Thus, for dies with large width the pressure drop will be large, inducing lip deformation known as clam shelling. This will also increase viscous heating and make the die not suitable for the extrusion of degradable material.

2 Alterations of the design equations of the Winter Die

It has been shown that the Winter Die, although designed with many desirable features, lacks practicality due to the large land length necessary to create extruded products of substantial width. Our challenge in working with the design equations of the Winter Die has been to maintain the desirable features of the die, while shortening the land length necessary to produce wide products. After careful analysis of the situation, we conclude that one alteration would combine relative simplicity and effectiveness; the manifold of constant width of the Winter Die will be replaced with a manifold whose width is dependent upon x. After changing w to $w(x)$, equation (1.21) becomes

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\left(\frac{b-x}{w(x)}\right) - 1}} \tag{2.1}$$

We will make the following assumptions concerning the manifold width and land length:

A.1 $y(b) = 0$

A.2 $w(0) = 1$

A.3 $w(x) = \frac{b-x}{n(x)}$

It follows from condition (A.3) above and equation (2.1) that

$$\frac{dy}{dx} = \frac{-1}{\sqrt{n(x) - 1}} \tag{2.2}$$

which upon integration gives

$$y(x) = -\int_b^x \frac{1}{\sqrt{n(x) - 1}}\, dx \tag{2.3}$$

and hence,

$$y(0) = -\int_0^b \frac{1}{\sqrt{n(x)-1}}\,dx \tag{2.4}$$

The function $n(x)$ has yet to be determined. In order to choose proper functions, we examine $n(x)$ more thoroughly. One requirement on $n(x)$ is

$$n(x) > 1 \qquad x \in [0, b] \tag{2.5}$$

We will consider the following two examples for $n(x)$.
Note. In order to show the validity of our results, a means for comparison must be created. Throughout research, the die width (b) was set equal to 50. Also, from assumptions (A.1) and (A.2), it is shown that the now varying manifold width ranges from one to zero. In order to capture a reasonable result for the maximum land length of the Winter die, we set $w = 0.5$ *in the equation for the maximum land length, giving a value of approximately 9.95 for a Winter die of width 50.*

Example 1: Polynomial

We chose to set

$$n(x) = s + rx^q \tag{2.6}$$

where $q > 0$. In this case we have

$$w(x) = \frac{b-x}{s+rx^q} \tag{2.7}$$

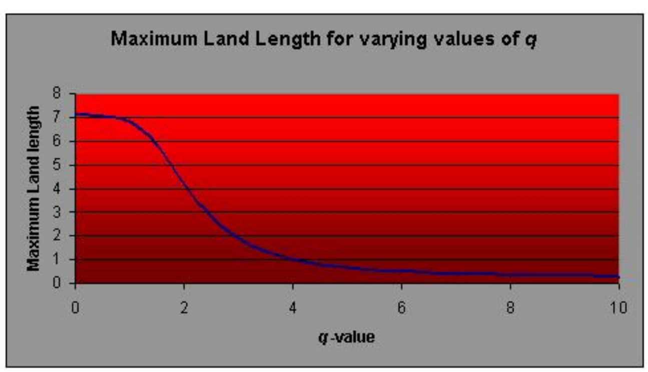
and

$$\frac{dy}{dx} = \frac{-1}{\sqrt{s+rx^q-1}} \tag{2.8}$$

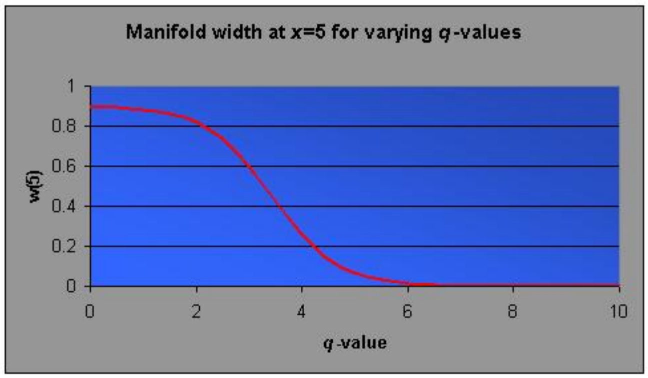
Integrating (2.8) and using boundary condition (A.1) we obtain the following forms of $w(x)$ and $y(x)$ corresponding to the values 0, 1, and 2 for q.

q	$w(x)$	$y(x)$
0	$w(x) = \frac{b-x}{s+r}$	$y(x) = \frac{b-x}{\sqrt{s-r-1}}$
1	$w(x) = \frac{b-x}{s+rx}$	$y(x) = \frac{2}{r}\left(\sqrt{rb+s-1} - \sqrt{rx+s-1}\right)$
2	$w(x) = \frac{b-x}{s+rx^2}$	$y(x) = \frac{\ln\left(\frac{\sqrt{r}b+\sqrt{rb^2+s-1}}{\sqrt{r}x+\sqrt{rx^2+s-1}}\right)}{\sqrt{r}}$

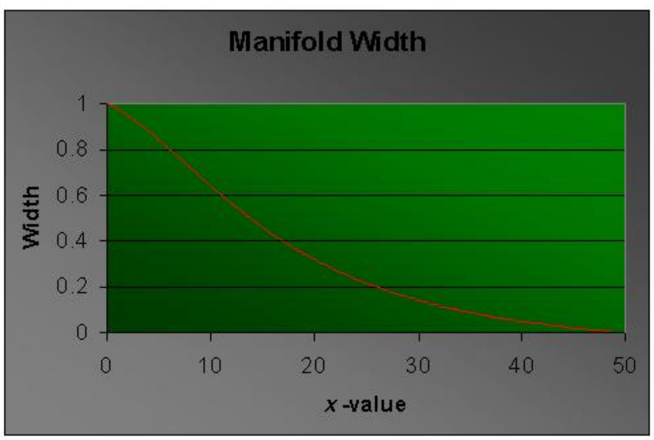
For $b = 50$, $r = .2$, and $s = 50$, we have the graph below, which shows that the value of y-max as q grows from zero to infinity approaches zero.



With q large, as x increases, the value of x^q becomes very large. Therefore, as x grows, $w(x)$ decreases rapidly, and this trend becomes increasingly prominent as q grows. This can be shown if we examine $w(5)$ for increasing values of q, as shown below (note: $w(0)$ has been set to 1).



By examining the two graphs previously shown, we must search for a q value that will not only decrease the maximum land length, but will also allow for a physically realistic manifold curve. By setting q equal to 1.8, a maximum land length of 4.8796 is attained (less than half the value for the Winter Die). The rate of change of the manifold (shown below) is not excessive, suggesting physical significance.



Example 2: Exponential

We choose $n(x)$ as follows:

$$n(x) = s - re^{qx} \tag{2.9}$$

which gives

$$w(x) = \frac{b - x}{s - re^{qx}} \tag{2.10}$$

and

$$\frac{dy}{dx} = \frac{-1}{\sqrt{s - re^{qx} - 1}} \tag{2.11}$$

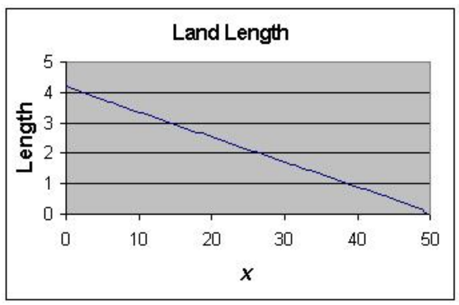
Integrating (2.10) with respect to x and using boundary condition (A.1) we obtain

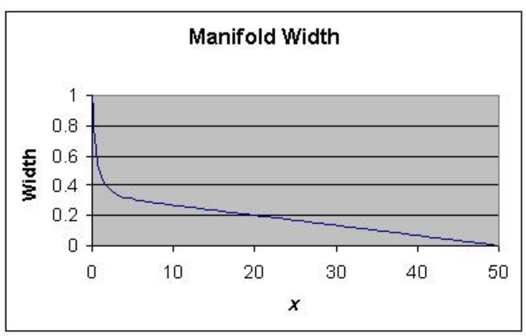
$$y(x) = \frac{2\left[\operatorname{arctanh}\left(\frac{\sqrt{s-re^{qx}-1}}{\sqrt{s-1}}\right) - \operatorname{arctanh}\left(\frac{\sqrt{s-re^{qb}-1}}{\sqrt{s-1}}\right)\right]}{q\sqrt{s-1}} \tag{2.12}$$

which gives

$$y_{\max} = y(0) = \frac{2\left[\operatorname{arctanh}\left(\frac{\sqrt{s-r-1}}{\sqrt{s-1}}\right) - \operatorname{arctanh}\left(\frac{\sqrt{s-re^{qb}-1}}{\sqrt{s-1}}\right)\right]}{q\sqrt{s-1}} \tag{2.13}$$

Again, it can be shown that by manipulating the parameters (such as by allowing r to approach zero) we can force the maximum land length to become zero. However, with these manipulations come problems similar to those found for the polynomial equation. By setting $q = -0.7$, $r = 100$, $s = 150$, and $b = 50$ we obtain the information represented below.





As can clearly be seen, if we adjust the parameters to attain only a moderate land length, the rate of change in the manifold is large. The linearity of the land length is suggestive that the exponential term may not be as effective as could be desired.

3 Concluding Remarks

1. As was assumed, $w(0)$ is given. By setting $w(0) = 1$, we force $b/s = 1$ for the first example and $b/(s-r) = 1$ for the second example. These conditions dictate the following:

   For example 1:
$$b = s \tag{3.1}$$
   and

   For example 2:
$$b = s - r \tag{3.2}$$

2. For example 2, as $q \to -\infty$, $y(0) \to 0$.

3. For example 2
$$\lim_{q \to 0} y(x) = \frac{b - x}{\sqrt{b - 1}} \tag{3.3}$$
   which shows that as q approaches zero, $y(x)$ and $w(x)$ become linear, which is equivalent to the situation that occurs in example 1 when $q = 0$.

4. For example 2, as we make q increasingly negative, the exponential term takes effect, and both $w(x)$ and $y(x)$ increase in curvature.

5. An important property in a uniform shear rate die is the equal residence time for each particle flowing through it. For the Winter Die, if we set
$$\frac{dy}{dx} = -\left[l(x) - 1\right]^{-1/2} \tag{3.4}$$
   where
$$l(x) = \frac{b - x}{w} \tag{3.5}$$
   then
$$\frac{da}{dx} = \left[\frac{l(x)}{l(x) - 1}\right]^{1/2} \tag{3.6}$$
   Also, the velocity in the manifold can be related to the velocity in the land as follows
$$v_m = l(x)^{1/2} v_s \tag{3.7}$$
   Using this relationship along with equation (3.7), the total residence time at x is given by
$$T(x) = \int_0^x \frac{1}{v_s(l(x) - 1)^{1/2}}\, dx + \frac{y(x)}{v_s} \tag{3.8}$$
   To show that the residence time is independent of x, it is enough to show that $T(x)$ is constant. To see this we notice that
$$\begin{aligned} \frac{dT(x)}{dx} &= \frac{1}{v_s(l(x) - 1)^{1/2}} + \frac{1}{v_s}\frac{dy}{dx} \\ &= \frac{1}{v_s(l(x) - 1)^{1/2}} - \frac{1}{v_s(l(x) - 1)^{1/2}} \\ &= 0 \end{aligned} \tag{3.9}$$
   for every x. Therefore, $T(x)$ is constant.

4 Acknowledgments

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