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IMPULSIVE DIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITIONS

Robert Knapik[1]

Department of Mathematics and Statistics, James Madison University, Harrisonburg, VA 22807.

In this article, we extend existence and uniqueness results of classical differential equations with initial conditions to some new type of equations called *impulsive differential equations with non-local conditions.*

The classical differential equation, given by

$$x'(t) = f(t, x(t)), \quad 0 \leq t \leq T, \quad x(0) = x_0, \quad x(t) \in \Re^n, \tag{1}$$

describes a system where the initial condition is given by $x(0) = x_0$, and a solution $x(t)$ is a continuous function under appropriate assumptions.

Recently, we have seen articles dealing with the equations where the systems are allowed to undergo some abrupt perturbations (harvesting, diseases, wars, etc.) whose duration can be negligible in comparison with the duration of the process. Therefore, in this case, a solution $x(t)$ may have jump discontinuities (to be called *impulses* for general equations) at times $t_1 < t_2 < \cdots$, given in the form of

$$x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i = 1, 2, \ldots \tag{2}$$

where I_i, $i = 1, 2, \ldots$, are some functions. (Of course, I_i may be identically zero, in which case there are no impulses.)

The differential equations incorporating jump discontinuities (steps) for their solutions are called *impulsive differential equations.* For example, Freedman, Liu and Wu [1991] studied models of single species growth with impulsive effect; Zavalishchin [1994] studied impulsive dynamic system for mathematical economics. See Rogovchenko [1997], Lakshmikantham, Bainov and Simeonov [1989] and Liu [1999] for more details.

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We have also seen articles dealing with *non-local conditions*. That is, the classical initial condition (also called "local condition") $x(0) = x_0$ is extended to the following non-local condition

$$x(0) + g(x(\cdot)) = x_0 \in \Re^n, \tag{3}$$

where $x(\cdot)$ is a solution and g is a mapping defined on some function space into $\Re^n$. (Of course, g may be identically zero, in which case it reduces to the local condition $x(0) = x_0$.) The advantage of using non-local conditions is that measurements at more places can be incorporated to get better models.

For example, for a non-uniform rod, $g(x(\cdot))$ may be given by

$$g(x(\cdot)) = \sum_{i=1}^{q} c_i x(s_i), \tag{4}$$

where c_i, $i = 1, \ldots, q$, are given constants and $0 < s_1 < s_2 < \cdots < s_q$. In this case, (4) allows the additional measurements at s_i, $i = 1, 2, \ldots, q$. A formula similar to (4) is also used in Deng [1993] to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In general, g may be an integral and may be non-linear. See Byszewski and Lakshmikantham [1990], and Lin and Liu [1996] for further studies of non-local conditions.

However, to our knowledge, we have not seen articles dealing with the combination of impulsive and non-local conditions; and there is some evidence that *impulsive differential equations with non-local conditions* should be investigated. For example, if a sound wave travels through a non-uniform rod (where non-local conditions can be applied), and if the wave's amplitude or frequency (parameter) changes in a piecewise continuous fashion with steps, then the vibration in the rod will also contain steps. So the merging of the impulsive and non-local conditions would be helpful in modeling this system.

Therefore, it is our purpose here to study the existence and uniqueness of solutions for the following impulsive differential equation with non-local conditions,

$$\begin{cases} x'(t) = f(t, x(t)), \quad 0 \le t \le T, \quad t \ne t_i, \\ x(0) + g(x(\cdot)) = x_0, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T, \end{cases} \tag{5}$$

in $\Re^n$, where f is a continuous function, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, and I_i's are some functions.

To make things more precise, we define the collection of piecewise continuous functions as

$$\begin{aligned} PC([0,T], \Re^n) \quad = \quad & \{x : x \text{ is a mapping from } [0,T] \text{ into } \Re^n \text{ such that } x(t) \text{ is} \\ & \text{continuous at } t \ne t_i \text{ and left continuous at } t = t_i, \text{ and} \\ & \text{the right limit } x(t_i^+) \text{ exists (finite) for } i = 1,\, 2,\, \ldots, p\}. \end{aligned}$$

Then, one can verify that $PC([0,T],\Re^n)$ is a Banach space under the norm

$$\|x\|_{PC} = \sup_{t\in[0,T]} |x(t)|, \tag{6}$$

where $|\cdot|$ is a norm in $\Re^n$.

Definition 1. *A* **solution** *of Eq.(5) is a function*

$$x(\cdot) \in PC([0,T],\Re^n) \cap C^1([0,T] \setminus \{t_1, t_2, \ldots, t_p\}, \Re^n),$$

which satisfies Eq.(5) on $[0,T]$.

Similar to the treatment for classical differential equations with initial conditions, let's reduce Eq.(5) to an integral form.

Theorem 1. *A function x in $PC([0,T],\Re^n)$ is a solution of Eq.(5) if and only if*

$$x(t) = [x_0 - g(x(\cdot))] + \int_0^t f(s,x(s))ds + \sum_{0<t_i<t} I_i(x(t_i)), \quad t \in [0,T]. \tag{7}$$

Proof. If x is a solution of Eq.(5), then for $t \in (t_j, t_{j+1}]$,

$$\begin{aligned}
\int_0^t f(s,x(s))\,ds &= \int_0^t x'(s)\,ds \\
&= \int_0^{t_1} x'(s)\,ds + \int_{t_1}^{t_2} x'(s)\,ds + \cdots + \int_{t_j}^{t} x'(s)\,ds \\
&= [x(t_1^-) - x(0^+)] + [x(t_2^-) - x(t_1^+)] + \cdots + [x(t^-) - x(t_j^+)] \\
&= [x(t_1^-) - x(0)] + [x(t_2^-) - x(t_1^+)] + \cdots + [x(t) - x(t_j^+)] \\
&= -x(0) - [x(t_1^+) - x(t_1^-)] - [x(t_2^+) - x(t_2^-)] - \\
&\qquad \cdots - [x(t_j^+) - x(t_j^-)] + x(t),
\end{aligned}$$

hence

$$\begin{aligned}
x(t) &= x(0) + \int_0^t f(s,x(s))\,ds + [x(t_1^+) - x(t_1^-)] + [x(t_2^+) - x(t_2^-)] + \\
&\qquad \cdots + [x(t_j^+) - x(t_j^-)] \\
&= x(0) + \int_0^t f(s,x(s))ds + \sum_{0<t_i<t} \Delta x(t_i) \\
&= [x_0 - g(x(\cdot))] + \int_0^t f(s,x(s))ds + \sum_{0<t_i<t} I_i(x(t_i)).
\end{aligned} \tag{8}$$

On the other hand, let $x(\cdot) \in PC([0,T],\Re^n)$ be a function satisfying Eq.(8). First, note that for this fixed function $x(\cdot)$, $g(x(\cdot))$ is a fixed element in $\Re^n$, and for $t \in (t_j, t_{j+1})$, $\sum_{0<t_i<t} I_i(x(t_i)) = \sum_{i=1}^{j} I_i(x(t_i))$ is a constant; thus $\frac{d}{dt}g(x(\cdot)) = 0$ and $\frac{d}{dt}\sum_{0<t_i<t} I_i(x(t_i)) = 0$ for $t \neq t_i$, $i = 1, 2, \dots, p$. Hence, we deduce that $x'(t) = f(t, x(t))$, $t \neq t_i$, $x(0) = x_0 - g(x(\cdot))$, and

$$\begin{aligned}\Delta x(t_i) &= x(t_i^+) - x(t_i^-) \\ &= \left[x(0) + \int_0^{t_i} f(s, x(s))ds + \sum_{j=1}^{i} I_j(x(t_j))\right] \\ &\quad - \left[x(0) + \int_0^{t_i} f(s, x(s))ds + \sum_{j=1}^{i-1} I_j(x(t_j))\right] \\ &= I_i(x(t_i)),\end{aligned}$$

which completes the proof. □

Hence, for x_0 fixed, (7) leads us to the definition of a mapping

$$P : PC([0,T], \Re^n) \to PC([0,T], \Re^n)$$

such that

$$(Px)(t) = [x_0 - g(x(\cdot))] + \int_0^t f(s, x(s))ds + \sum_{0<t_i<t} I_i(x(t_i)). \tag{9}$$

Based on this, we list the following conditions so that the Contraction Mapping Principle can be applied to establish the existence and uniqueness for Eq.(5).

(H). $f : [0,T] \times \Re^n \to \Re^n$, $g : PC([0,T], \Re^n) \to \Re^n$, and $I_i : \Re^n \to \Re^n$, $i = 1, 2, \dots, p$, are continuous functions and there exist constants $L > 0,\ K > 0,\ h_i > 0,\ i = 1, 2, \dots, p$, such that

$$|f(t,u) - f(t,v)| \leq K|u - v|,\ t \in [0,T],\ u,\ v \in \Re^n, \tag{10}$$

$$|g(x(\cdot)) - g(y(\cdot))| \leq L\|x(\cdot) - y(\cdot)\|_{PC},\quad x(\cdot),\ y(\cdot) \in PC([0,T], \Re^n), \tag{11}$$

$$|I_i(u) - I_i(v)| \leq h_i|u - v|,\ u,\ v \in \Re^n, \tag{12}$$

and that

$$L + KT + \sum_{i=1}^{p} h_i < 1. \tag{13}$$

Theorem 2. *Let Assumption (H) be satisfied. Then for every $x_0 \in \Re^n$, Eq.(5) has a unique solution on $[0,T]$, satisfying*

$$x(t) = [x_0 - g(x(\cdot))] + \int_0^t f(s, x(s))ds + \sum_{0<t_i<t} I_i(x(t_i)). \tag{14}$$

Proof. Let $x_0 \in \Re^n$ be fixed, and consider the mapping $P : PC([0,T], \Re^n) \to PC([0,T], \Re^n)$ defined by (9). Then we have, for $v,\ w \in PC([0,T], \Re^n)$,

$$
\begin{aligned}
&|(Pv)(t) - (Pw)(t)| \\
&\quad \leq \quad |g(v(\cdot)) - g(w(\cdot))| + \int_0^t |f(s, v(s)) - f(s, w(s))|\, ds \\
&\qquad\qquad + \sum_{0<t_i<t} |I_i(v(t_i)) - I_i(w(t_i))| \\
&\quad \leq \quad L\|v(\cdot) - w(\cdot)\|_{PC} + K \int_0^t |v(s) - w(s)|\, ds \\
&\qquad\qquad + \sum_{0<t_i<t} h_i |v(t_i) - w(t_i)| \\
&\quad \leq \quad L\|v(\cdot) - w(\cdot)\|_{PC} + KT\|v(\cdot) - w(\cdot)\|_{PC} + \left(\sum_{0<t_i<t} h_i \right) \|v(\cdot) - w(\cdot)\|_{PC} \\
&\quad \leq \quad \left\{ L + KT + \sum_{i=1}^{p} h_i \right\} \|v(\cdot) - w(\cdot)\|_{PC}, \quad t \in [0,T],
\end{aligned}
\tag{15}
$$

or

$$
\|Pv - Pw\|_{PC} \leq \left\{ L + KT + \sum_{i=1}^{p} h_i \right\} \|v - w\|_{PC}. \tag{16}
$$

Now, from Assumption (H), we conclude that P is a contraction mapping on $PC([0,T], \Re^n)$. Therefore, the Contraction Mapping Principle can be applied to obtain a unique fixed point for the mapping P, which, according to Theorem 1, gives rise to a unique solution of Eq.(5) on $[0,T]$. This completes the proof. □

Remark. Theorem 2 includes the classical differential equations with initial conditions (that is, without impulsive and non-local conditions) as special cases of Eq.(5), in which $L = 0$ and $h_i = 0$, $i = 1, 2, \ldots, p$, and hence (13) reduces to $KT < 1$, which is the requirement using the Contraction Mapping Principle for classical differential equations with initial conditions.

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